IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X.
PP 25-27
www.iosrjournals.org

Some Categorical Aspects of C-Spaces-I

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Abstract. *In this paper some simple categorical aspects of c-spaces are at- tempted. Separators and co-separators are characterized.*

I. Introduction

In 1983, Reinhard Börger proposed the theory of connectivity spaces, which ruled out the shortfalls in the use of the theory of connected sets in practical cases. This concept generalized the concept of connectedness in both topology and graph theory. He proved that the category of connectivity spaces *Zus* over *Set* is a topological category which is not cartesian closed. A systematic study of this space was carried out by J. Serra[9] and further extended by H. J. A. M. Hejimans[3], C. Ronse [6] etc. This space found profound applications in the areas of *Image Segmentation, Image Filtering, Image Coding, Digital Topology, Pattern Recognition, Mathematical Morphology etc* [3, 4, 7, 8, 9]. An initiative towards the mathematical study of the structure of c-spaces can be found in [1, 2, 5]. In this paper we attempt to give a concrete proof to some basic concepts that follows from the concept of Topological Category.

II. Preliminaries

A c-space[5] is a set X together with a collection C of subsets such that the following properties hold.

(1) $\varphi \in C$ and $\{x\} \in C$ for every $x \in X$.

(ii) If $\{C_i : i \in I\}$ be a non empty collection of subsets in C with $\bigcap_{i \in I} C_i \neq \varphi$ then $\bigcup_{i \in I} C_i \in C$.

Other terminologies used for c-spaces are *connectivity space*[1, 9] and *integral con- nectivity space*[2]. In his paper [1], Reinhard Börger have the same definition for the connectivity space except that empty set is connected. The collection C of subsets X which satisfies (*i*) and (*ii*) is called a *c-structure* [5] or a *connectivity class* [3, 4, 9]

2010 *Mathematics Subject Classification.* 54A05, 54D05,05C10, 05C40.
Key words and phrases. c-space, c-continuous function,category .
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of X. We like to follow the terminology used in [5]. Elements of a c-structure are called *connected sets*.

$D_X = \{\varphi\} \cup \{\{x\} : x \in X\}$ is a c-structure on X, called the *discrete c-structure* and the space (X, D_X) is called the *discrete c-space*. Also, $I_X = P(X)$, the power set of X is a c-structure on X, called the *indiscrete c-structure* and the correspond- ing space is called an *indiscrete c-space*. The c-space (X, C_X) is denoted by X if there is no ambiguity.

Let X and Y be two c-spaces and $f : X \to Y$ be a function. f is called *c-continuous* [5] or *catenuous* [5] or a *connectivity morphism*[3, 9] or a connectivity map[2], if it maps connected sets of X to connected sets of Y.

A function $f : X \to Y$ is said to be a quotient map if Y has the smallest c-structure with respect to which f is c-continuous. In this context, Y is s said to be the quotient of X with respect to f.

All categorical terminologies are taken from the text book of Strecker[10]. For definitions, readers are requested to refer the same.

III. On some Basics of the Category of c-Spaces

It can be noted that [1, 2] collection of c-spaces with c-continuous functions as morphisms forms a category. We like to denote the category of c-spaces by *Cnc*. By *Mor*(A, B), we mean the collection of c-continuous functions from A to B. It can be noted that *Cnc* is neither thin nor discrete and not small.

Proposition 3.1. *An object A is an initial object in Cnc if and only if it is the empty c-Space.*
Proof. Let A be an initial object in Cnc . Then $|Mor(A, B)| = 1$ for every c-space B. If $A \neq \varphi$, then $|Mor(A, B)| > 1$ for every indiscrete c-space B with $|B| > 1$, a contradiction to the choice of A. Thus $A = \varphi$ and hence A is an empty c-space.
Conversely let A be an empty c-space. Since we can define exactly one function from empty set to any set, it is obvious that A is an initial object in Cnc . Q

Proposition 3.2. *An object A is a terminal object in Cnc if and only if* $|A| = 1$.
Proof. Let A be a terminal object in Cnc . Then by the definition, $|Mor(B, A)| = 1$ for every c-space B. If $|A| > 1$, then $|Mor(B, A)| > 1$ for every discrete c-space B with $|B| > 1$, a contradiction to the choice of A. Hence $|A| \leq 1$. Since A cannot be φ, we have $|A| = 1$.
Conversely let A be a c-space with $|A| = 1$. Since $|A| = 1$, the only c-morphism from B to A is the constant map. Hence $|Mor(B, A)| = 1$ for every c-space B.
Thus A is a terminal object. Q

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Corollary 3.3. *Cnc has no zero objects.*
Proposition 3.4. *In Cnc, the separators are precisely the nonempty c-spaces.*

Proof. Let the c-spaces A and B be such that $A \overset{f}{\underset{g}{\rightrightarrows}} B$ are distinct c-morphisms.
Then there exists $a \in A$ such that $f(a) \neq g(a)$. Let S be any nonempty Define $h : S \longrightarrow A$ by $h(x) = a$, for all $x \in S$. Being a constant map, h is a c-morphism. It can be easily verified that $(f \circ h)(x) = f(a)$ and $(g \circ h)(x) = g(a)$ for all $x \in S$. Since $f(a) \neq g(a)$, we have $f \circ g \neq g \circ h$. Hence S is a separator. Q
Proposition 3.5. *In Cnc , indiscrete spaces with at least two points is a co- separator.*

Proof. Let $B \overset{f}{\underset{g}{\rightrightarrows}} A$ be two morphisms such that $f(b) \neq g(b)$ for some $b \in B$. Let D be an indiscrete c-space with at least two elements c_1 and c_2.
Define $h: A \longrightarrow D$ by $h(x) = \begin{cases} c_1 & if\ x = f(b) \\ c_2 & elsewhere \end{cases}$
Clearly h is a c-morphism with $h \circ f \neq h \circ g$. Hence D is a co-separator. Q
Remark 3.6. *The above condition is not necessary.*

Proof. Consider the c-space A and B where $A = \{a,b,c\}$, $C_A = D_A \cup \{\{a,b\},\{b,c\},\{a,b,c\}\}$, $B = \{1,2\}$ and $C_B = D_B \cup \{\{1,2\}\}$.
Define $f : B \to A$ as $f(1) = a$, $f(2) = b$ and $g : B \to A$ as $g(1) = b$, $g(2) = a$. Here f and g are c-morphisms such that $f \neq g$. Choose a c-space E with $E = \{d, e, f\}$ and $C_E = D_E \cup \{\{d, e\}, \{e, f\}, \{d, e, f\}\}$.
Define $h : A \to E$ as $h(a) = d$, $h(b) = e$ and $h(c) = f$. Then h is a c-morphism with $h \circ f \neq h \circ g$. Hence E is a co-separator. But note that E is not an c-space. Q
Theorem 3.7. *Let $A, B \in$ Cnc with $f, g \in Mor(B, A)$ with $f \neq g$. Then X is a co-separator of f and g if and only if X contains the quotient space of A with respect to some c-morphism $h \in Hom(A, X)$ with $h \circ f \neq h \circ g$.*
Proof. Let $f, g \in Mor(B, A)$ with $f \neq g$. Let X be a co-separator of f and g. Then there exits a c-morphism $h : A \to X$ such that $h \circ f \neq h \circ g$. Since h is a c-morphism, C_X should contain the c-structure $C_A^* = \langle \{h(C) : C \in C_A\} \rangle$. Obviously $(h(A), C_A^*)$ is the quotient space of A with respect to h.

Conversely let X contains the quotient space of A with respect to some function $h : A \to X$ with $h \circ f$ $f = h \circ g$. Then h is a c-morphism with $h \circ f$ $f = h \circ g$. Hence X is a co-separator of f and g. Q

Acknowledgement

The author is very much indebted to his supervising guide Dr.P.T. Ramachan- dran, Department of Mathematics, University of Calicut for introducing me to the area of c-spaces.

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